# Math 564: Advance Analysis 1
## Lecture 5

Since the cylinders generate the Borel $\sigma$-algebra of $2^{\mathbb{N}}$ and the boxes generate the Borel $\sigma$-alg. of $\mathbb{R}^d$, we get the so called Bernoulli$(p)$ measure $\mu_p$, $p \in (0,1)$, and the Lebesgue measure on $\mathbb{R}^d$, defined on all Borel sets.

**Def.** For a topol. space $X$, the Borel $\sigma$-algebra $\mathcal{B}(X)$ is that generated by open sets. The sets in $\mathcal{B}(X)$ are called Borel sets. A Borel measure on $X$ is any measure defined on $\mathcal{B}(X)$.

In particular, the Bernoulli$(p)$ and Lebesgue are Borel measures.

An example of non-unique extension of premeasure. Let $\mathcal{A}$ be the algebra generated by the half-intervals $[a,b) \subseteq \mathbb{R}$, i.e. $\mathcal{A}$ consists of finite unions of half-intervals of the form $[a,b)$, where we treat $(-\infty, a)$ as a half interval. Define a premeasure $\mu$ on $\mathcal{A}$ by setting

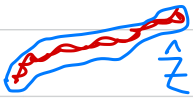
$$\mu(A) := \begin{cases} \infty & \text{if } A \neq \emptyset \\ 0 & \text{o.w.} \end{cases}$$

Then $\langle \mathcal{A} \rangle_\sigma = \mathcal{B}(\mathbb{R})$ and $\mu^*(\text{any nonempty set}) = \infty$. Here are two other extensions to $\mathcal{B}(\mathbb{R})$: for each $B \in \mathcal{B}(\mathbb{R})$, put

$$\mu_1(B) := \begin{cases} 0 & \text{if } B \text{ is ctbl,} \\ \infty & \text{o.w.} \end{cases}$$

$$\mu_2(B) := \text{the counting measure} = \begin{cases} \infty & \text{if } B \text{ is infinite} \\ |B| & \text{o.w.} \end{cases}$$

**Measurable sets.** Let $(X, \mathcal{B})$ be a measurable space, i.e. $\mathcal{B}$ a $\sigma$-alg on $X$. Let $\mu$ be a measure on $\mathcal{B}$. We call $(X, \mathcal{B}, \mu)$ a *measure space*.
A set $Z \subseteq X$ is called *$\mu$-null* if $\exists\, \hat{Z} \in \mathcal{B}$ s.t. $Z \subseteq \hat{Z}$ and $\mu(\hat{Z}) = 0$.

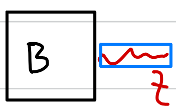



Let $\mathrm{Null}_\mu$ denote the collection of $\mu$-null sets.

**Obs.** $\mathrm{Null}_\mu$ is a $\sigma$-ideal, i.e. it contains $\emptyset$ and is closed under subsets and ctbl unions.

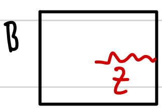
**Lemma.** $\forall\, B \in \mathcal{B}$ and $Z \in \mathrm{Null}_\mu$,
(a) $B \cup Z = \tilde{B} \setminus \tilde{Z}$ for some $\tilde{B} \in \mathcal{B}$ and $\tilde{Z}$ $\mu$-null.
(b) $B \setminus Z = \tilde{B} \cup \tilde{Z}$ for some $\tilde{B} \in \mathcal{B}$ and $\tilde{Z}$ $\mu$-null.

**Proof.** (a)




Let $\hat{Z} \in \mathcal{B}$ with $\hat{Z} \supseteq Z$ and is $\mu$-null. Then set $\tilde{B} := B \cup \hat{Z} \in \mathcal{B}$ and $\tilde{Z} := \hat{Z} \setminus Z$.

(b)




... $\square$

Let $\mathrm{Meas}_\mu := \{ B \cup Z : B \in \mathcal{B},\ Z \in \mathrm{Null}_\mu \}$, call the sets in this *$\mu$-measurable*.

**Prop.** $\mathrm{Meas}_\mu$ is the $\sigma$-algebra generated by $\mathcal{B} \cup \mathrm{Null}_\mu$.
**Proof.** It's enough to show that $\mathrm{Meas}_\mu$ is a $\sigma$-algebra.
- Complements: Let $B \cup Z \in \mathrm{Meas}_\mu$, then $(B \cup Z)^c = B^c \cap Z^c = B^c \setminus Z$ $\in \mathrm{Meas}_\mu$ by the lemma above.
- Ctbl unions: Let $B_n \cup Z_n \in \mathrm{Meas}_\mu$, then $\bigcup_{n \in \mathbb{N}} (B_n \cup Z_n) = \left(\bigcup_{n \in \mathbb{N}} B_n\right) \cup \left(\bigcup_n Z_n\right) = B \cup Z \in \mathrm{Meas}_\mu$. $\square$

**Prop.** The measure $\mu$ admits a unique extension to $\mathrm{Meas}_\mu$, called the *completion* of $\mu$, denoted $\bar{\mu}$.
**Proof.** Define $\bar{\mu}$ on $\mathrm{Meas}_\mu$ by setting $\bar{\mu}(B \cup Z) := \mu(B)$ for $B \in \mathcal{B}$, $Z \in \mathrm{Null}_\mu$.

We show that $\bar{\mu}$ is well-defined: let $M = B_0 \cup Z_0 = B_1 \cup Z_1$, where $B_0, B_1 \in \mathcal{B}$ and $Z_0, Z_1 \in \mathrm{Null}_\mu$. Need to show $\mu(B_0) = \mu(B_1)$.
Let $B := B_0 \cap B_1$ and let $\hat{Z} \supseteq Z_0 \cup Z_1$ be a $\mu$-null Borel set. Then

$$\mu(B) \le \mu(B_i) \le \mu(B \cup \hat{Z}) = \mu(B)$$

because $B_i \subseteq B \cup \hat{Z} = B \cup Z_0 \cup Z_1$ because $B_0 \setminus B_1 \subseteq Z_1$ and $B_1 \setminus B_0 \subseteq Z_0$.
Thus, $\mu(B_i) = \mu(B) = \mu(B_{1-i})$.
For uniqueness, let $\nu$ be any extension to $\mathrm{Meas}_\mu$. Then

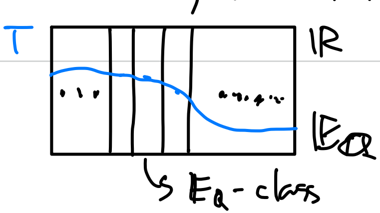
$$\mu(B) = \nu(B) \le \nu(B \cup Z) \le \mu(B \cup \hat{Z}) = \mu(B), \text{ so } \nu(B \cup Z) = \mu(B),$$

where $B, \hat{Z} \in \mathcal{B}$, $Z, \hat{Z}$ are null and $\hat{Z} \supseteq Z$. □

We will drop bar from the notation $\bar{\mu}$ and just write $\mu$ for the completion as well.

**Remark.** Note that any Polish space has a ctbl open basis, like rational open boxes in $\mathbb{R}^d$, which implies that there are only continuum many open sets, hence also only continuum many Borel sets. However, because $\mathcal{P}(\text{any null set}) \subseteq \mathrm{Meas}_\mu$ and some measures, like Lebesgue and Bernoulli$(p)$, have continuum-sized null sets, we get that $|\mathrm{Null}_\mu|$ can be $2^{\text{continuum}}$. So typically there are many-many more $\mu$-measurable sets than Borel sets.

**An example of a non-measurable set.** We'll construct a non-Lebesgue measurable subset of $\mathbb{R}$. Let $E_\mathbb{Q}$ be the coset equivalence relation of $\mathbb{Q} \le \mathbb{R}$, i.e. $x \, E_\mathbb{Q} \, y :\iff x - y \in \mathbb{Q}$. Each $E_\mathbb{Q}$-class is ctbl (it's a copy of $\mathbb{Q}$), so there are continuum-many classes. Using Axiom of Choice, we get a transversal $T$ for $E_\mathbb{Q}$, i.e. a set that



intersects every $\mathbb{E}_{\mathbb{Q}}$-class in exactly one point.

Claim. $T_1 := T \cap [0,1)$ is not Lebesgue measurable.
"You shall never pick a point from each class" — D. Gaboriau.

Proof. Suppose $T_1$ is Lebesgue measurable.
Note that for $q_0 \neq q_1$ rationals, $q_0 + T_1$ and $q_1 + T_1$ are disjoint, so
$$[0,1] \subseteq \bigsqcup_{\substack{q \in \mathbb{Q} \\ \cap [-1,1]}} (q + T_1) \subseteq [-1, 2].$$
Because Lebesgue measure is translation invariant, $\lambda(q+T_1) = \lambda(T_1)$ $\forall q \in \mathbb{Q}$. Thus,
$$1 = \lambda([0,1)) \leq \sum_{q \in \mathbb{Q} \cap [-1,1)} \lambda(q+T_1) = \sum_{q \in \mathbb{Q} \cap [-1,1]} \lambda(T_1) \leq \lambda([-1,2]) = 3.$$

If $\lambda(T_1) = 0$, then $1 \leq 0$, and if $\lambda(T_1) > 0$, then $\infty \leq 3$, a contradiction. $\square$

Remark. (a) The Lebesgue measure $\lambda$ on $\mathbb{R}^d$ is translation invariant, i.e. $\forall x \in \mathbb{R}^d$ and $A \subseteq \mathbb{R}^d$ $\lambda$-measurable,
$$\lambda(A) = \lambda(x + A).$$
This is because it's true for boxes by definition.

(b) The Bernoulli$(p)$ measure $\mu_p$, $p \in (0,1)$, on $2^{\mathbb{N}}$ is shift-invariant, where the shift is the transformation $s: 2^{\mathbb{N}} \to 2^{\mathbb{N}}$
$$(x_n)_{n \in \mathbb{N}} \mapsto (x_{n+1})_{n \in \mathbb{N}}.$$
Being shift invariant means:
$s^{-1}(A)$ has the same measure as $A$, for any meas. $A \subseteq 2^{\mathbb{N}}$.
This comes from the condition that $\mathrm{Prob}[x \in A] = \mathrm{Prob}[s(x) \in A] = \mathrm{Prob}[x \in s^{-1}(A)]$. Shift-invariance is true because it's true for cylinders:
$s^{-1}[w] = [0w] \sqcup [1w]$, so $\mu_p(s^{-1}[w]) = \mu_p([0w]) + \mu_p[1w] = (1-p) \cdot \mu_p([w]) + p \cdot \mu_p([w]) = \mu_p([w])$.